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THE VIRTUAL SOLVABILITY OF THE FUNDAMENTAL GROUP OF A GENERALIZED LORENTZ SPACE FORM

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Introduction

Let $\mathrm{Aff}_n(\mathbb{R})$ denote the group of all affine transformations of the real affine vector space $\mathbb{R}^n$. It is well known that $\mathrm{Aff}_n(\mathbb{R})$ is isomorphic to the semidirect product $\mathrm{Gl}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ is identified with the group of all translations of $\mathbb{R}^n$. Let $\pi\colon \mathrm{Aff}_n(\mathbb{R}) \to \mathrm{Gl}_n(\mathbb{R})$ be the natural projection. A subgroup $\Gamma \subset \mathrm{Aff}_n(\mathbb{R})$ is called G-linear if $\pi(\Gamma) \subset G$, where G is a real algebraic group, i.e., G is the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points of an algebraic subgroup $\mathbf{G}$ of $\mathrm{Gl}_n(\mathbb{C})$ defined over $\mathbb{R}$. Let G^0 be the connected component of G, and let $G^0 = SR$ be the Levi decomposition of G^0, where R is the solvable radical of G, and S is a maximal semisimple subgroup of G^0. Let $S = S_1 S_2 \cdots S_r$ be an almost direct product of simple Lie subgroups S_i. The group Γ is called a group of generalized Lorentz motions if every S_i is a group of (real) rank $\mathrm{rk}_{\mathbb{R}} S_i \le 1$. (By a rank of S_i we mean the dimension of any maximal $\mathbb{R}$-split torus in the Zariski closure $\mathbf{S}_i$ of S_i in $\mathbf{G}$.) Assume that Γ acts properly discontinuously on $\mathbb{R}^n$ (i.e., the set $\{\gamma \in \Gamma | \gamma K \cap K \neq 0\}$ is finite for every compact $K \subset \mathbb{R}^n$), and that the quotient $\mathbb{R}^n/\Gamma$ is compact. In the case where Γ is a group of Lorentz motions (that is $G = \mathrm{SO}(n-1, 1)$) it was proved in [9] that Γ is a virtually solvable group, i.e., Γ contains a solvable subgroup of finite index. The aim of the present paper is to prove similar results for all groups Γ of generalized Lorentz motions.

Theorem A. *Let Γ be a G-linear subgroup of $\mathrm{Aff}_n(\mathbb{R})$. Assume that* (a) *$\Gamma$ acts properly discontinuously on $\mathbb{R}^n$*, (b) *$\mathbb{R}^n/\Gamma$ is compact, and* (c) *Γ is a group of generalized Lorentz motions. Then Γ is a virtually solvable group.*

According to a result of G. A. Margulis [15] if Γ is a group of generalized Lorentz motions which acts properly discontinuously on $\mathbb{R}^n$ but $\mathbb{R}^n/\Gamma$ is not compact, then Γ is not necessarily a virtually solvable group.

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Also, remark that by the recent result of Y. Carrière [6, Theorem 1.2.1] $\pi^{-1}(\pi(\Gamma)\cap\overline{\pi(\Gamma)}^0)$, where $\overline{\pi(\Gamma)}^0$ is the connected component of the closure of $\pi(\Gamma)$ in the Euclidean topology of $\mathrm{GL}(n, R)$, is a unipotent group. Furthermore, [17, Proposition 3.10] implies that Γ is a virtually solvable group if and only if it is a virtually polycyclic group.

Theorem A has a natural geometrical reformulation. Recall that an affine manifold is one which admits a covering by a coordinate system where the overlap homeomorphisms should extend to affine transformations from $\mathrm{Aff}_n(\mathbb{R})$. An affine space form M is a compact affine manifold which is also geodesically complete, i.e., the universal covering manifold M is affinely diffeomorphic to $\mathbb{R}^n$. It is well known (cf. [26, Corollary 1.9.6]) that any affine space form is obtained by forming the quotient $\mathbb{R}^n/\Gamma$ of $\mathbb{R}^n$ by a subgroup $\Gamma\subset\mathrm{Aff}_n(\mathbb{R})$ which acts on $\mathbb{R}^n$ freely, properly discontinuously, and with compact fundamental domain. If Γ is G-linear and $\mathrm{rk}_{\mathbb{R}} S_i \le 1$ for every simple factor S_i of the semisimple part S of G^0, then we shall call M a generalized Lorentz space form (compare [9]). Since every finitely generated linear group contains a torsion free subgroup of finite index (a theorem of Selberg, see [19]) Theorem A is equivalent to the following.

Theorem B. *The fundamental group $\pi_1(M)$ of a generalized Lorentz space form is virtually solvable. In particular, M has a finite covering diffeomorphic to a solvmanifold.*

Our theorem affirms (for generalized Lorentz space forms) a long standing conjecture due to L. Auslander (see [1], [15], [16]) that the fundamental group of any affine space form is virtually solvable. Except for Lorentz space forms [9] (see, also, the work [7] for $n = 4$), Theorem A (equivalently, Theorem B) has been established in the following particular cases: (a) for euclidean space forms, i.e., Γ is a discrete subgroup of $\mathrm{O}(n)\ltimes\mathbb{R}^n$ (this is the classical Bieberbach theorem [17, Corollary 8.26]); (b) when $n = 2, 3$ [8]; and (c) when G is a reductive group and, furthermore, $\mathrm{rk}_{\mathbb{R}} G \le 1$, the result was recently proved by F. Grunewald and G. A. Margulis [10].

In a somewhat weaker form our result was previously proved in [24]. Finally, note that a slight modification of our method proves the abovementioned conjecture for small values of n (at least for $n = 4, 5$ [25]).

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1. On the action of solvable radicals on $\mathbb{R}^n$

In this section Γ is a subgroup of $\mathrm{Aff}_n(\mathbb{R})$ such that Γ acts properly discontinuously on $\mathbb{R}^n$, and $\mathbb{R}^n/\Gamma$ is compact. Since $\mathrm{Aff}_n(\mathbb{R})$ is a real algebraic group, we can consider the Zariski closure H of Γ in $\mathrm{Aff}_n(\mathbb{R})$. Let R (resp. U) be the solvable (resp. unipotent) radical of H.

The following general fact is well known [20]. If Γ is a discrete group of automorphisms of a contractible manifold X, and Γ acts freely on X, then $\operatorname{cd}\Gamma \leq \dim X$, where $\operatorname{cd}\Gamma$ stands for the cohomological dimension of Γ; the equality $\operatorname{cd}\Gamma = \dim X$ holds if and only if X/Γ is compact. Recall that if Γ is any discrete group, and $\Gamma' \subset \Gamma$ is a torsion free group of finite index, then the virtual cohomological dimension $\operatorname{vcd}\Gamma$ of Γ is equal to $\operatorname{cd}\Gamma'$ [20].

1.1 Lemma.[1] *With the above notation and assumptions, U acts transitively on $\mathbb{R}^n$.*

Proof. The group H is a semidirect product $M \ltimes U$ of its reductive subgroup M and the unipotent radical U. Since $\mathrm{Aff}_n(\mathbb{R})$ can be viewed as a subgroup of $\mathrm{Gl}_{n+1}(\mathbb{R})$ acting on a hyperplane $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$ (see, for example, [2]), the action of M on $\mathbb{R}^n$ admits a fixed point $x_0 \in \mathbb{R}^n$. Hence $Hx_0 = Ux_0$. It is well known that Ux_0 is closed in $\mathbb{R}^n$ and homeomorphic to a real vector space [4], [18]. On the other hand, Γ acts properly discontinuously on Ux_0, and the quotient Ux_0/Γ is compact. In view of the Selberg theorem [19], Γ contains a subgroup of finite index, which acts freely on $\mathbb{R}^n$. Therefore $\operatorname{vcd}\Gamma = \dim Ux_0 = \dim \mathbb{R}^n$, i.e., $Ux_0 = \mathbb{R}^n$. The lemma is proved.

1.2. Let $\Lambda = \Gamma \cap R$ and let R_1 be the Zariski closure of Λ in H. In view of Lemma 1.1 the group H acts transitively on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $h \in H$, then $h(R_1x) = (hR_1h^{-1})hx = R_1(hx)$. Therefore H acts transitively on the set $X = \{R_1x | x \in \mathbb{R}^n\}$ of all orbits of R_1. In particular, X can be identified with the homogenous space H/H_0, where H_0 is the isotropy group of an element $y_0 \in X$, and H acts on H/H_0 by left multiplications. But H contains a maximal reductive subgroup M fixing y_0 and $H = M \ltimes U$. Therefore H/H_0 is isomorphic (as a real algebraic variety) to U/U_0, where $U_0 = U \cap H_0$. Since U/U_0 is isomorphic to a real vector space we obtain the following.

Lemma. *X can be identified with a real vector space in such a way that the real algebraic group H acts algebraically on X.*

[1] This result was first proved by W. Goldman and M. W. Hirsch, Trans. Amer. Math. Soc. **295** (1986) 175–198 (Theorem 2.6).

1.3. Lemma. *Let $Y = R_1x$, $x \in \mathbb{R}^n$, and U_1 be the maximal unipotent subgroup of R_1. Then*

(a) *U_1 acts transitively and freely on Y,*
(b) *Λ acts properly discontinuously on Y and Y/Λ is compact.*

In particular, Λ is a finitely generated group.

Proof. Let $R_1 = T_1U_1$, where T_1 is a reductive subgroup of R_1. There is a point $x_0 \in \mathbb{R}^n$ such that $R_1x_0 = U_1x_0$. Since U_1 is a normal subgroup of H,

$$R_1(hx_0) = h(R_1x_0) = h(U_1x_0) = U_1(hx_0)$$

for every h in H. Hence U_1 acts transitively on Y, and the reductive subgroup T_1 can be chosen in such a way that $T_1x = x$. It is well known that every discrete Zariski dense subgroup of $\mathbb{R}^k$ is a cocompact lattice in $\mathbb{R}^k$. Using this one easily proves (by induction on $\dim R_1$ and reducing to the case when R_1 is abelian) that $R_1 = \Lambda K T_1$ for some compact $K \subset R_1$. Hence (b) is proved. In order to prove (a) we assume the contrary, i.e., let $ux = x$ for some $u \in U_1$, $u \neq 1$. Denote $Y' = R_1/T_1$. There is a continuous R_1-equivariant map $\varphi: Y' \to Y$. Let $\varphi x' = x$, $x' \in Y'$. For every positive integer i, let $u^ix' = \lambda_ic_ix'$, where $\lambda_i \in \Lambda$ and $c_i \in K$. It is easy to see that $\{\lambda_i | i \in \mathbb{N}\}$ must be an infinite subset of Λ. On the other hand, $x = u^ix = \varphi(\lambda_ic_ix') = \lambda_ic_ix$. Since Γ acts properly discontinuously on $\mathbb{R}^n$, the set of pairwise different λ_i must be finite. This contradiction completes our proof.

1.4. We set $\widetilde{\Gamma} = \Gamma/\Lambda$ and consider the action of $\widetilde{\Gamma}$ on X.

Proposition. (a) *$\widetilde{\Gamma}$ acts properly discontinuously on X, and the quotient $X/\widetilde{\Gamma}$ is compact,*

(b) $\dim X = \operatorname{vcd} \widetilde{\Gamma}$,
(c) $\operatorname{vcd} \Gamma = \operatorname{vcd} \Gamma/\Lambda + \operatorname{vcd} \Lambda$.

Proof. Assume (a) holds. Then (b) and (c) follow directly from Lemma 1.2 and Lemma 1.3, respectively.

Let us prove (a). The compactness of $X/\widetilde{\Gamma}$ follows from the compactness of $\mathbb{R}^n/\Gamma$. The rest of the proof will be given in several steps. Let $\Lambda' = R_1^0 \cap \Lambda$, where R_1^0 is the connected component of the identity in R_1. First note that there is a connected Lie subgroup L of R_1 such that $\Lambda' = L \cap \Lambda$ and L/Λ' is compact. This follows easily by induction on $\dim R_1$ (reducing to the case where R_1 is abelian) from the following facts: (1) the commutator group $[R_1, R_1]$ is a real unipotent algebraic group, (2) $\Lambda \cap [R_1, R_1]$ is cocompact in $[R_1, R_1]$, (3) Λ is a finitely generated group, and (4) the simply connected covering of $R_1^0/[R_1, R_1]$ is homeomorphic to a real vector space. Next note that $Lx = R_1 x$ for each $x \in \mathbb{R}^n$. To see this fix a maximal reductive subgroup $T_1 \subset R_1$ with $T_1 x = x$. The group R_1 is a semidirect product of U_1 and T_1. Let $\varphi: R_1 \to U_1$ be the natural projection. It is enough to prove that for each $u \in U_1$ there exists a $g \in L$ such that $\varphi(g) = u$. But L contains $[R_1, R_1]$ and, therefore, we can reduce the proof to the trivial case where R_1 is abelian.

In order to finish the proof of (a) we fix a compact $C \subset L$ such that $L = \Lambda' C$. Let $\psi: \Gamma \to \widetilde{\Gamma}$ be the natural homomorphism and $\theta: \mathbb{R}^n \to X$ be the factor map (i.e., $\theta(x) = U_1 x$ for every $x \in \mathbb{R}^n$). Let $K \subset X$ be a compact set. Fix a compact $K' \subset \mathbb{R}^n$ with $\theta(K') = K$. Let $\tilde{\gamma} \in \widetilde{\Gamma}$ and $\tilde{\gamma} K \cap K \neq \varnothing$. Then $\gamma(CK') \cap CK' \neq \varnothing$ for some $\gamma \in \Gamma$ with $\psi(\gamma) = \tilde{\gamma}$. But $\{\gamma \in \Gamma | \gamma(CK') \cap CK' \neq \varnothing\}$ is a finite set. Therefore $\widetilde{\Gamma}$ acts properly discontinuously on X. The proposition is proved.

1.5. Denote $H_1 = H/R$. The group $\widetilde{\Gamma}$ can be embedded in H_1. Note that $\widetilde{\Gamma}$ is a discrete subgroup of H_1. This easily follows from the following theorem due to L. Auslander (see [17, Theorem 8.24]). Let G be a Lie group, R be a connected solvable normal subgroup of G, and $\psi: G \to G/R$ be the natural homomorphism. Then if Γ is a discrete subgroup of G, the connected component $\overline{\psi(\Gamma)}^0$ of the identity in the closure of $\psi(\Gamma)$ in G/R is solvable.

Let K be a maximal compact subgroup of H_1. Then the quotient H_1/K (called the symmetric space of H_1) is known to be homeomorphic to a real vector space. The group $\widetilde{\Gamma}$ acts (by left multiplications) on H_1/K. Since $\widetilde{\Gamma}$ is discrete in H_1, the action of $\widetilde{\Gamma}$ on H_1/K is properly discontinuous. Hence $\operatorname{vcd} \widetilde{\Gamma} \le \dim H_1/K$. Now Proposition 1.4 implies the following.

Corollary. $\dim X \le \dim H_1/K$.

2. Proof of Theorem A

2.1. Let N be the kernel of the action of H on X. Then N is a normal algebraic subgroup of H. Denote $\widetilde{H} = H/N$. In view of Selberg's theorem we may, and we will, assume that $\widetilde{\Gamma}$ acts freely on X. Since $R_1 \subset N$ we obtain that $\widetilde{\Gamma}$ embeds in $\widetilde{H}$. On the other hand, $\widetilde{\Gamma}$ acts properly discontinuously on X, and $\widetilde{H}$ acts continuously on X. Therefore $\widetilde{\Gamma}$ is a discrete subgroup of $\widetilde{H}$.

2.2. Let $\widetilde{R}$ (resp. $\widetilde{U}$) be the solvable (resp. unipotent) radical of $\widetilde{H}$. Also, denote by $\widetilde{S}$ (resp. $\widetilde{P}$) a maximal semisimple (resp. reductive) subgroup of $\widetilde{H}$. We assume that $\widetilde{P} \supset \widetilde{S}$. In view of the definition of X (see 1.2) there is a point $a \in X$ which is fixed by $\widetilde{P}$. Let V be the tangent space of X at a, and $\rho: \widetilde{P} \to \mathrm{Gl}(V)$ be the representation of $\widetilde{P}$ on V. Since H acts algebraically and faithfully on X, the representation ρ is faithful (see [3]).

Lemma. *If $x \in \widetilde{P}$, then $\rho(x)$ has an eigenvalue equal to* 1.

Proof. Let $\varphi: \widetilde{H} \to \widetilde{P}$ be the projection of $\widetilde{H}$ on $\widetilde{P}$. (Recall that $\widetilde{H} = \widetilde{P} \ltimes \widetilde{U}$ is a semidirect product of $\widetilde{P}$ and $\widetilde{U}$.) Since $\widetilde{\Gamma}$ is Zariski dense in $\widetilde{H}$, it is enough to prove that $\rho\varphi(\gamma)$ has an eigenvalue 1 for each $\gamma \in \widetilde{\Gamma}$. Let $\gamma = \gamma_s\gamma_u$ be the Jordan decomposition of γ in $\widetilde{H}$ [5, Chapter 1]. There is an element $u \in \widetilde{U}$ such that $\gamma_s \in u\widetilde{P}u^{-1}$. Let A be the smallest (unipotent) algebraic subgroup of $\widetilde{H}$ containing γ_u. Denote $b = ua$. Since γ_s commutes with every element from A and $\gamma_s b = b$, we get that γ_s fixes the orbit Ab pointwise. On the other hand, the orbit Ab is γ-invariant. In view of Proposition 1.4(a) Ab is homeomorphic to a vector space $\mathbb{R}^k$ with $k > 0$.

Let $\gamma_s = uhu^{-1}$, $h \in \widetilde{P}$. It is easy to see that h fixes $(u^{-1}A)b$ pointwise. Since $(u^{-1}A)b$ is homeomorphic to a nontrivial vector space and $a \in (u^{-1}A)b$, we obtain that $\rho(h)$ has an eigenvalue equal to 1. Note that $\gamma_s = h(h^{-1}uhu^{-1})$ and $h^{-1}uhu^{-1} \in \widetilde{U}$. This implies $\varphi(\gamma_s) = h$. On the other hand, the homomorphism $\rho\varphi$ preserves the Jordan decomposition (cf. [5, Theorem 4.4]). Therefore $\rho\varphi(\gamma)$ has an eigenvalue equal to 1. The lemma is proved.

2.3. The above lemma, inspired by a conversation with G. A. Soifer, is designed to replace the strong Jung-van de Kulk theorem (see [12], [14]) in the initial version [24] of our proof. Since the use of this theorem seems to be important from a conceptual point of view, we recall its formulation and sketch how it can be applied to the present situation. Let $\mathrm{GA}_2(\mathbb{C})$ be the group of all regular automorphisms of the affine space $\mathbb{C}^2$. An automorphism φ of $\mathbb{C}^2$ is given by $f = (f_1, f_2)$, where f_1 and f_2 are polynomials from $\mathbb{C}[x_1, x_2]$. (The $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x_1, x_2]$ corresponding to φ sends $x_i \to f_i$, $i = 1, 2$.) The triangular ("Borel") subgroup of $\mathrm{GA}_2(\mathbb{C})$ is $\mathrm{BA}_2(\mathbb{C}) = \{f \in \mathrm{GA}_2(\mathbb{C}) | f_1 = a_1x_1 + b_1,\ f_2 = a_2x_2 + p(x_1)\}$, where $a_1 \in \mathbb{C}^*$, $b_1 \in \mathbb{C}$, and $p(x_1) \in \mathbb{C}[x_1]$. The theorem of Jung-van der Kulk says that $\mathrm{GA}_2(\mathbb{C})$ is generated by its subgroups

$\mathrm{Aff}_2(\mathbb{C})$ and $\mathrm{BA}_2(\mathbb{C})$. I. R. Shafarevich [22] (see also the later work [13]) proved that $\mathrm{GA}_2(\mathbb{C}) = \mathrm{Aff}_2(\mathbb{C}) *_C \mathrm{BA}_2(\mathbb{C})$ is the amalgamated free product with $C = \mathrm{Aff}_2(\mathbb{C}) \cap \mathrm{BA}_2(\mathbb{C})$. Note that $\mathrm{BA}_2(\mathbf{C})$ and $\mathrm{GA}_2(\mathbb{C})$ are infinite dimensional algebraic groups in the sense of [22]. On the other hand, if $L \subset \mathrm{GA}_2(\mathbb{C})$ is a finite-dimensional algebraic subgroup of $\mathrm{GA}_2(\mathbb{C})$, then in view of a result of J.-P. Serre [21] L is conjugated to a subgroup of $\mathrm{Aff}_2(\mathbb{C})$ or $BA_2(\mathbb{C})$. It is easy to see that if L is a subgroup of $\mathrm{BA}_2(\mathbb{C})$, then it is solvable.

Now we return to our particular situation and assume that $\dim X = 2$ and $\widetilde{S} = \mathrm{SL}_2(\mathbb{R})$. (It follows from 2.5 below that this is the case when the rank of S does not exceed 1.) We consider the complexification $\widetilde{\mathbf{H}}$ of $\widetilde{H}$ and the action of $\widetilde{\mathbf{H}}$ on $X \otimes \mathbb{C}$ $(\cong \mathbb{C}^2)$. Now it easily follows from the above results and the solution of the Auslander conjecture for $n = 2$ [8, §2] that $\widetilde{\Gamma}$ (equivalently, Γ) is a virtually solvable group.

2.4. Next we need the following result. (See [10, Proposition 2.6] for an independent proof.)

Lemma. *Let Q be a simple (real) algebraic group and $\mathrm{rk}_{\mathbb{R}} Q \leq 1$. Let d be the dimension of the minimal (nontrivial) representation of Q, and s be the dimension of the symmetric space of Q. Then $d \geq s$ and $d = s$ if and only if Q is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and $d = s = 2$.*

Proof. Let $\mathbf{Q}$ be an $\mathbb{R}$-simple algebraic group (i.e., $\mathbf{Q}$ does not contain any proper infinite normal algebraic subgroup defined over $\mathbb{R}$) such that $\mathbf{Q}(\mathbb{R}) = Q$. Assume that $\mathbf{Q}$ is not an absolutely simple algebraic group (i.e., $\mathbf{Q}$ is not a simple algebraic group over $\mathbb{C}$). The algebraic group $\mathbf{Q}$, admits a simply connected covering $\widetilde{\mathbf{Q}}$ defined over $\mathbb{R}$ [23, Proposition 2.6.1]. There exists an algebraic group $\mathbf{P}$ defined over $\mathbb{C}$ such that $\widetilde{\mathbf{Q}} = R_{\mathbb{C}/\mathbb{R}}\mathbf{P}$, where $R_{\mathbb{C}/\mathbb{R}}\mathbf{P}$ is the restriction of $\mathbf{P}$ to $\mathbb{R}$ [23, 3.1.2]. But $\mathrm{rk}_{\mathbb{R}} S = \mathrm{rk}_{\mathbb{R}} \widetilde{\mathbf{S}} = \mathrm{rk}_{\mathbb{C}} \mathbf{P} = 1$, where $\mathrm{rk}_{\mathbb{C}} \mathbf{P}$ is the rank of the algebraic group $\mathbf{P}$ over $\mathbb{C}$. Therefore $\mathbf{P} = \mathrm{SL}_2(\mathbb{C})$ and $\widetilde{\mathbf{S}}(\mathbb{R})$ is homeomorphic to $\mathrm{SL}_2(\mathbb{C})$. Hence the symmetric space of S is homeomorphic to $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$. So, if $\mathbf{Q}$ is not an absolutely simple algebraic group, then $d = 4$ and $s = 3$. Assume that $\mathbf{Q}$ is an absolutely simple algebraic group. It follows from the classification results in [11] (or [23]) that Q is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(1, n)$ $(n \geq 2)$, $\mathrm{SO}(1, n)$ $(n \geq 4)$, $\mathrm{Sp}(1, n)$ $(n \geq 2)$, and a group of type F_4 with rank 1. Let H be one of those groups, and d' be the dimension of its standard representation. It is well known that d' does not exceed the dimension of the minimal representation of any group H' locally isomorphic to H. Now our lemma follows from the following information about s and d' extracted from [11, Table 5, p. 518]:

Symmetric space	s	d'
$SL_2(\mathbb{R})/SO(2)$	2	2
$SU(1, n)/S(U_1 \times U_n)$	$2n$	$2(n+1)$
$SO(1, n)^0/SO(n)$	n	$n+1$
$Sp(1, n)/Sp(1) \times Sp(n)$	$4n$	$4(n+1)$
$(f_{4(-20)}, SO(9))$	16	26

The lemma is proved.

2.5. Let $\widetilde{S}$ be a maximal semisimple subgroup of $\widetilde{H}$. Assume that $\widetilde{S} \neq \{1\}$, and let $\widetilde{S} = \widetilde{S}_1\widetilde{S}_2\cdots\widetilde{S}_\gamma$ be an almost direct product of simple real algebraic groups $\widetilde{S}_i$. Let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition of the tangent space V (see 2.2) on irreducible S-submodules. It is well known that every V_i is a tensor product of irreducible nontrivial $\widetilde{S}_j$-modules V_{ij}, i.e., $V_i = \bigotimes_{j=1}^{r_i} V_{ij}$, $r_i \in \mathbb{N}$. Let $n_{ij} = \dim_{\mathbb{R}} V_{ij}$ and $n = \dim_{\mathbb{R}} V$. Then $n = \sum_{i=1}^{k}(\prod_{j=1}^{r_i} n_{ij})$. For every $\widetilde{S}_j$ let d_j be the dimension of the minimal (nontrivial) real representation of $\widetilde{S}_j$, and let s_j be the dimension of the symmetric space of $\widetilde{S}_j$. Then $s = s_1 + \cdots + s_r$ will be the dimension of the symmetric space of S. Since ρ is a faithful representation,

$$n \geq d_1 + d_2 + \cdots + d_r.$$

In view of Corollary 1.5 we have

$$s_1 + s_2 + \cdots + s_r \geq n \geq d_1 + d_2 + \cdots + d_r.$$

According to Lemma 2.4, $d_i \geq s_i$. Therefore $d_i = s_i = 2$ for every $i = 1, 2, \cdots, r$ (Lemma 2.4). Hence every V_{ij} is a standard $SL_2(\mathbb{R})$-module. In particular, there is an element $x \in \widetilde{S}$ such that $\rho(x)$ does not have an eigenvalue equal to 1. The latter contradicts Lemma 2.2. Our Theorem A is proved.

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